by Richard O. Michaud

The Markowitz Optimization Enigma: Is 'Optimized' Optimal?

The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with MV optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained MV optimization can yield results that are inferior to those of simple equal-weighting schemes.

Nevertheless, MV optimization is superior to many ad hoc techniques in terms of integration of portfolio objectives with client constraints and efficient use of information. Its practical value may be enhanced by the sophisticated adjustment of inputs and the imposition of constraints based on fundamental investment considerations and the importance of priors. The operating principle should be that, to the extent that reliable information is available, it should be included as part of the definition of the optimization procedure.

THE MARKOWITZ MEAN-variance (MV) efficient frontier is the standard theoretical model of normative investment behavior.[1] Most modern finance textbooks consider mean-variance efficiency the method of choice for optimal portfolio construction and asset allocation and as a means for rationalizing the value of diversification. The Markowitz efficient frontier has also provided the basis for many important advances in positive financial economics, including the Sharpe-Lintner Capital Asset Pricing Model (CAPM) and recognition of the fundamental dichotomy between systematic and diversifiable risk.[2]

Given the success of the efficient frontier as a conceptual framework, and the availability for nearly 30 years of a procedure for computing efficient portfolios, it remains one of the outstanding puzzles of modern finance that MV optimization has yet to meet with widespread acceptance by the investment community, particularly as a practical tool for active equity investment management. Does this "Markowitz optimization enigma" reflect "a considered judgment [by the investment community] that such methods are not worthwhile," or is it "merely another case of deep-seated resistance to change."[3] The enigma is not easily dismissed by targeting the inadequate training in contemporary finance and mathematics of many practicing investment professionals. There are simplified MV-optimization procedures that are neither mathematically cumbersome nor anti-intuitive.[4]

This article demonstrates that the enigma can be rationalized in many instances. The traditional MV procedure often leads to financially irrelevant or false "optimal" portfolios and asset allocations. In fact, equal weighting can be shown to be superior to MV optimization in some cases.[5] However, new techniques address some of the limitations of traditional MV optimizers, improving the practical investment value of portfolio optimization.[6]

1. Footnotes appear at end of article.

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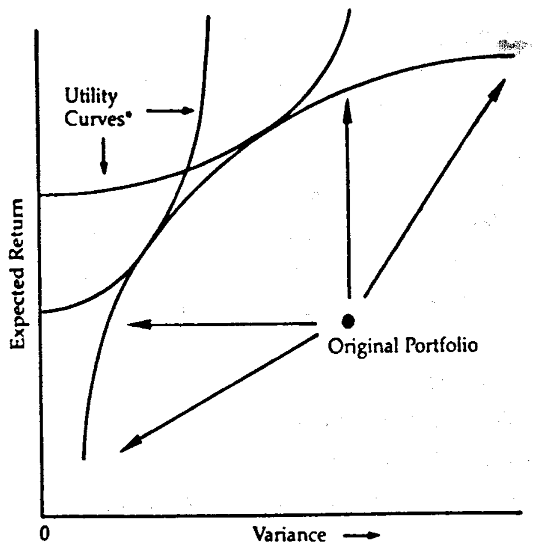
Classical Markowitz MV Optimization

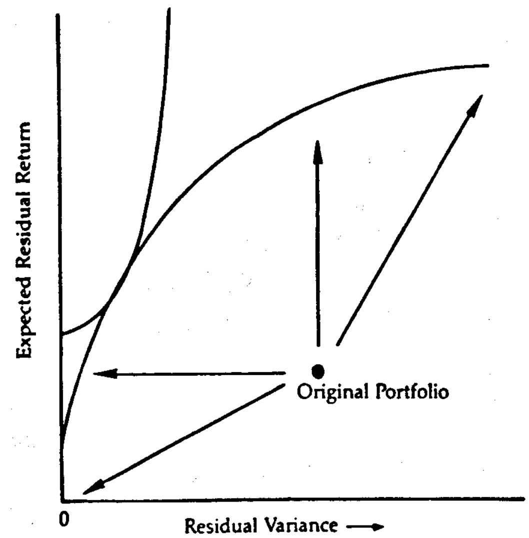
Classical MV optimization assumes that the investor prefers a portfolio of securities that offers maximum expected return for some given level of risk (as measured by the variance of return). Given estimated means, standard devi-

Figure A Markowitz Mean-Variance Efficient Frontier



*Maximum expected utility indifference curves.

Figure B Expected Residual Return-Risk MV Efficient Frontier



ations and correlations of return for N securities, the MV-optimization procedure selects the proportions of investable wealth to devote to each security. The resulting set of prescribed portfolio weights (X_1 through X_n) describe optimal solutions. (See the appendix for a mathematical formulation.)

The set of optimal portfolios for all possible levels of portfolio risk defines the MV efficient frontier. Figure A illustrates the classical MV efficient frontier in terms of the mean-variance of total return. Except for the names attached to the securities, the problems of equity portfolio optimization and asset allocation are equivalent in this framework. Figure B illustrates the efficient frontier in terms of the mean-variance of residual return, or "alpha"; in the case of equity portfolio optimization, alpha is usually defined as return in excess of the rate of return associated with the security's assumed systematic risk.[7] The optimal portfolio for any particular investor is the portfolio on the efficient frontier that is tangent to the "utility curve" that defines that investor's relative risk aversion.

In many cases of practical interest, the efficient frontier is defined subject to a budget constraint (sum of the proportions of invested wealth equal to one) and no short-selling (non-negative proportions of invested wealth). Other linear constraints, including trading costs, may be imposed. Computing the MV efficient frontier requires (parametric) quadratic programming.

Benefits of MV Optimizers

This article focuses on the limitations of MV optimizers, but it is important to keep in mind some of the significant potential benefits of the technology. These are outlined below.

- *Satisfaction of client objectives and constraints*: Portfolio optimizers provide a convenient framework for integrating a wide variety of simple but important client constraints and objectives with portfolio structure.
- *Control of portfolio risk exposure*: Portfolio optimizers can be used to control the portfolio's exposure to various components of risk.
- *Implementation of style objectives and market outlook*: An organization's investment style, philosophy and market outlook may be reflected within the MV framework by choice of the appropriate exposure to various risk factors, the stock universe of interest and the performance benchmark.
- *Efficient use of investment information*: Optimizers are designed to use information op-

timally in a total portfolio context, while *ad hoc* weighting can be counterproductive with respect to available information.[8]

- *Timely portfolio changes*: Portfolio optimizers can process large amounts of information quickly, a particularly important benefit for a large institution, which needs to determine the impact of new information on all its portfolios quickly and conveniently.

Simple Reasons for Not Using MV Optimizers

Against these benefits of MV optimization there are arrayed some simple (though not necessarily robust) reasons for *not* using optimization. Optimizers make conceptual demands on portfolio managers. As products of modern finance, portfolio optimizers may seem difficult to use because investment managers are used to a more informal tradition of investment management. Optimization depends, explicitly or implicitly, on specification of an appropriate benchmark or normal portfolio, which may reflect a manager's investment style, philosophy or outlook. Consequently, information is reflected, not by the exclusion or inclusion of securities in the portfolio, but by the extent to which portfolio weights deviate from normal or benchmark weights.

Probably the single most important reason why many financial institutions don't use portfolio optimizers is political. This is because the effective use of an optimizer mandates significant changes in the structure of the organization and the management of the investment process. In many investment organizations the investment policy committee, which often consists of the senior officer(s) of the firm, makes the key investment decisions. An optimizer may tend to usurp many of the integrative functions of the committee.

Introduction of an optimizer will also tend to encourage the development of a more quantitative investment process, which may involve unwelcome adjustments. For one thing, it will increase significantly the level of accountability, communication and risk-sharing within the organization. This is because quantitative valuation models require that input forecasts be stated explicitly, while the valuation process itself provides return estimates that are unambiguous descriptions of value. For another, control of the optimization algorithm requires a working knowledge of basic statistical concepts and modern portfolio theory. As the ability to understand the financial meaning of the statistical characteristics of a portfolio becomes critical, quantitatively-oriented specialists inevitably assume a central role in the investment process. It is therefore not very surprising that traditional managers of large financial institutions are not eager to relinquish their positions of power and influence by allowing an optimizer and a quantitative specialist to usurp key roles in the investment process.

Organizational politics or inexperience with modern financial technology cannot fully explain the Markowitz optimization enigma. If MV optimizers added value, new investment management firms, organized and staffed to manage and leverage the technology, would eventually displace more traditional firms.

It is known anecdotally that a number of experienced investment professionals have experimented with MV optimizers only to abandon the effort when they found their MV-optimized portfolios to be unintuitive and without obvious investment value. As a practical matter, even absent the influence of organizational politics, the optimized portfolios were often found to be unmarketable either internally or externally.

Some Fundamental Limitations

The key operative issue in regard to MV optimizers can be stated simply, in terms of two alternative hypotheses:

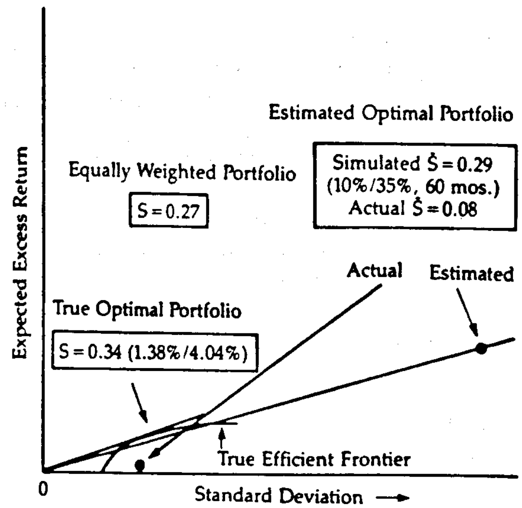
(1) MV-optimized portfolios are better, even though they are difficult to understand.
(2) MV-optimized portfolios are difficult to understand because they don't make investment sense and don't have investment value.

This article argues that the unintuitive character of MV-optimized portfolios is often symptomatic of the absence of significant investment value. MV optimizers have serious financial deficiencies, which will often lead to financially meaningless "optimal" portfolios.

Error Maximization

The unintuitive character of many "optimized" portfolios can be traced to the fact that MV optimizers are, in a fundamental sense, "estimation-error maximizers." Risk and return estimates are inevitably subject to estimation error. MV optimization significantly over-

Figure C The Error-Maximization Effect



Source: Data from J.D. Jobson and B. Korkie "Putting Markowitz Theory to Work," *Journal of Portfolio Management*, Summer 1981.

weights (underweights) those securities that have large (small) estimated returns, negative (positive) correlations and small (large) variances. These securities are, of course, the ones most likely to have large estimation errors.

Jobson and Korkie have quantified the magnitude of the error-maximization characteristics of MV optimizers in certain cases.[9] Using a known multivariate distribution of monthly returns for 20 stocks, they found the "optimal" portfolio, defined as that portfolio on the efficient frontier with the maximum Sharpe ratio (excess return divided by the standard deviation). Then, using Monte Carlo simulations, they estimated expected returns, variances and covariances for the 20 stocks over a 60-month period and computed the "optimal" portfolio for each set of estimates. Finally, they compared the true Sharpe ratios of (1) the average of the simulated optimal portfolios; (2) the optimal portfolio derived from the known multivariate distribution; and (3) an equally weighted portfolio of the 20 stocks. The true Sharpe ratios were, respectively, 0.08, 0.34 and 0.27! Their results, illustrated in Figure C, dramatically confirm the error-maximization hypothesis.

One caveat should be noted for accurate interpretation of the Jobson-Korkie results: The computed optimal portfolios did not include a short-selling constraint. Including this condition would have reduced the magnitude of the differences across the Sharpe ratios of the three portfolios. Furthermore, most financial institutions do have short-selling constraints. The Jobson-Korkie conclusions thus need to be moderated, although they are not invalidated, when applied to a realistic investment management setting.[10] These results also strongly confirm the importance of imposing financially meaningful constraints, when they are available, on the MV-optimization procedure.

A practical and general consequence of the error-maximization process is that any estimates of the statistical characteristics of optimized portfolios, if those characteristics are part of the optimization objective function, may be significantly biased. The measure of diversifiable risk produced by the optimizer, for example, is likely to be a significant underestimate of the optimal portfolio's true level of risk.

Good and Bad Estimators

An important contributor to the error-maximizing character of MV optimization when using historical data is that the usual estimation procedure—which replaces expected returns with their sample means—is (generally) not optimal.

An estimator is "admissible" if there exists no other estimator that dominates it for a given risk or loss function.[11] Stein has shown that, under standard conditions, sample means are not an admissible estimator of expected returns.[12] Intuitively, sample means are suboptimal because they ignore the inherent multivariate nature of the problem. More powerful statistical estimation techniques are required.

Missing Factors and Non-Financial Structure

MV optimization often ignores factors that are fundamentally important investment management considerations. One of the most important of these factors is liquidity, or the percentage of a company's market capitalization represented by portfolio holdings.

A portfolio of a large bank trust department or a portfolio of small-cap stocks, for example, may hold a significant percentage of a security's market capitalization. A 1 per cent change in the portfolio may thus represent a very substantial amount of the total value of the firm. As the proportion of the total value of the company purchased (sold) by the portfolio becomes significant, the purchase (sale) price is likely to rise (fall).

Figure D MV Frontier with Liquidity Factor

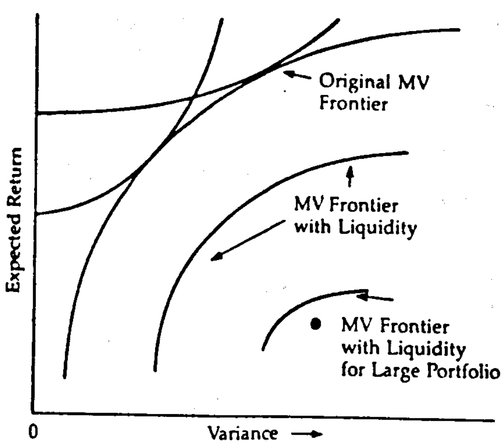


Figure D illustrates the impact of liquidity on the set of efficient portfolios. Compared with the traditional (unconstrained) MV frontier, imposition of a liquidity constraint results in less return enhancement and/or less risk reduction. In an extreme case, for very large capitalization portfolios, the MV frontier may be close to the original, "unoptimized," portfolio.

Liquidity considerations suggest some important principles for valid portfolio optimization. (1) For a large financial institution, the "true" optimal portfolio may not differ substantially from its current portfolio. (2) Large financial institutions must necessarily use care in revising their portfolios. (3) An optimal portfolio is inherently position-dependent, even when traditional transaction costs are ignored.

Mismatched Levels of Information

Optimizers do not differentiate between levels of uncertainty associated with the inputs. This problem is not confined to the difference in uncertainty between return and risk estimates; there are also significant differences across the levels of uncertainty associated with input estimates for various classes of stocks, such as utilities versus growth stocks.

A related problem is that, in many cases, differences in estimated means may not be statistically significant. In such cases, the primary value of MV analysis may be to reduce portfolio risk.[13]

Unstable Optimal Solutions

In some cases, MV optimizations are highly unstable; that is, small changes in the input assumptions can lead to large changes in the solutions. One important reason for this behavior is ill-conditioning of the covariance matrix. MV optimization requires the inversion of a covariance matrix; an ill-conditioned matrix will generally result in unstable solutions. Input assumptions that do not reflect financially meaningful estimates or the use of parameter estimates based on insufficient historical data are often associated with ill-conditioning and instability.

Non-Uniqueness

Optimizers, in general, produce a unique "optimal" portfolio for a given level of risk. This appearance of exactness is highly misleading, however. The uniqueness of the solution depends on the erroneous assumption that the inputs are without statistical estimation error.

As Figure E illustrates, given any point on the true MV efficient frontier, there is a neighborhood of the point (illustrated by the shaded area on and below the frontier) that includes an infinite number of statistically equivalent portfolios.[14] These "optimally equivalent" portfolios may have significantly, even radically, different

Figure E Statistically Equivalent MV Efficient Portfolios

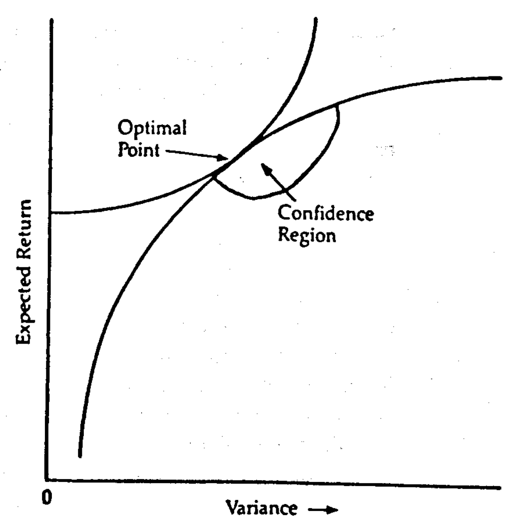
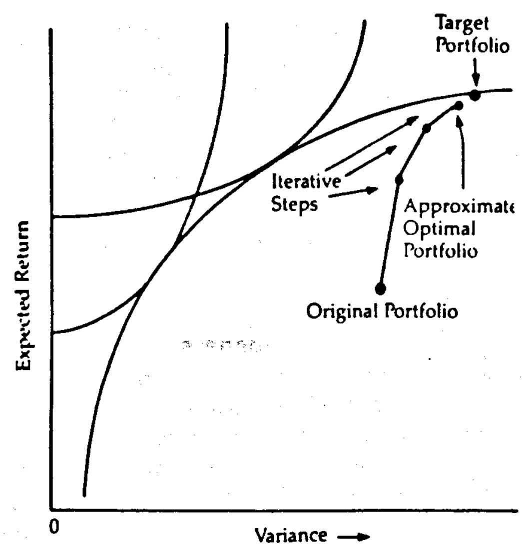


Figure F Approximate-Estimate MV Efficient Frontier



portfolio structures. In effect, this means that optimal portfolio structure is fundamentally not well defined.

Exact vs. Approximate MV Optimizers

A variety of commercially available optimization algorithms are marketed as MV optimizers.[15] Some provide "exact" (quadratic programming), others "approximate" optimal solutions. The difference determines such characteristics as (a) processing time; (b) entire frontier vs. single-point solution; (c) maximum size of the optimization universe; and (d) the ability to operate on standard personal computers.

Quadratic (parametric) programming, a generalization of linear programming, can solve for the entire MV (or alpha-diversifiable risk) efficient frontier. The primary limitations of the procedure are a relatively small universe size and/or relatively long computational time. Although the algorithm can include transaction costs, they have often been ignored. Enhancements of the traditional algorithm, which allow for the solution of relatively large-scale optimization problems in the presence of factor models of risk, are available.[16]

Approximate MV optimizers are able to solve optimizations for "institutional-size" portfolios and generally include transaction costs and options to optimize other factors. Their important limitation is that they provide a single "optimal" portfolio that is "near" the MV efficient frontier. As Figure F shows, this procedure finds a succession of "more optimal" portfolios at each iteration, ceasing the search when a portfolio within a specified tolerance is found.

Some practitioners have claimed that MV-optimal portfolios derived from approximate optimizers don't seem particularly unintuitive. Do the characteristics of approximate and exact optimizers differ? Are approximate optimizers "better" in some fundamental, practical sense? Or are they only better at hiding the limitations of MV optimization? If both procedures solve for the same objective function under the same constraints, the results should be identical. Some reasons for any observed differences are discussed below.

Inadequate Approximation Power

Approximate MV-optimal portfolios, because they *are* approximations, reflect less of the information in the input estimates, including the effects of error maximization, than exact optimizers.

This problem may be particularly acute for many PC-based optimization programs. Often used for asset allocation studies, PC optimization programs can reflect their lack of approximation power by a remarkable lack of instability in their solutions. Very different input assumptions have led to similar "optimal" allocations.[17]

Default Settings of the Parameters

Approximate optimizers solve for a single optimal portfolio near the MV efficient frontier. To single out an optimal portfolio, they must assume some value for the "target" parameter in the objective function. Risk-aversion or suitability parameters may have been set to target the more "explainable" parts of the MV frontier. At the maximum-return end of the efficient frontier, an MV-optimized portfolio has a very easily definable structure—maximize return (or alpha) and ignore risk.

Approximate MV optimizers *are* convenient. But the benefit of convenience should be weighed against the additional and unpredictable level of error they impose on the optimization process. Except in cases where it is computationally infeasible, parametric quadratic programming remains the algorithm of choice.

Enhancing MV Optimization

Procedures for enhancing MV optimization share important similarities. They are often Bayesian in character and depend on the existence of a prior, either for adjustment of the inputs or as a constraint on the optimization.

Asset Allocation With Respect to a Benchmark

While it is common institutional practice to define equity portfolio optimization in terms of a benchmark such as a market index, it is infrequent in asset allocation studies. Yet in many cases MV efficiency is properly defined in terms of performance with respect to an index or a liability. (In the context of a liability benchmark, the problem is sometimes called "surplus" management.)

The MV-optimization procedure for funding a liability is called "benchmark asset allocation."[18] Liability changes represent the prior for judging the benefits of returns associated with an asset allocation in a given period. The optimization's input parameters are redefined to reflect residual returns with respect to the benchmark.

Introducing a benchmark can significantly alter the characteristics of MV-optimal asset allocations. In general, benchmark asset allocation is strongly dependent on the economic characteristics of the liabilities. It is consequently far less time-period dependent or unstable and appears to have substantially more practical investment value.

Bayes-Stein Shrinkage Estimators

Bayes-Stein estimators constitute an important class of admissible estimators of expected returns when historical data are used. Observed sample means for individual assets are "shrunk" to some global mean. The global mean may represent the pooled mean, a Bayesian prior or the mean of the minimum-variance efficient-frontier portfolio. In many cases, the greater the variability in the historical data, the greater the shrinkage of sample means to the global mean.[19] Tests of Bayes-Stein estimation have shown that it can improve traditional MV optimization significantly.[20]

For a given utility function, Bayes-Stein estimation generally changes the MV-optimal portfolio, shrinking the recommended optimal mix in the direction of the minimum-variance efficient portfolio.[21] The amount of shrinkage of the input estimates generally increases as the number of asset classes increases and the number of time periods decreases. Bayes-Stein estimators represent an important emerging technology with significant potential for improving the practicality of MV optimization.

The IC Adjustment

In many cases, financial institutions use stock valuation models, rather than historical returns, to estimate the "returns" input to an optimizer. Such "estimates" generally take the form of relative valuations, rankings or simple ordinal assignments.

The optimizer requires a ratio-scale estimate of return or alpha for each security so that the return and risk input estimates, usually derived from historical data, are on comparable scales. Ambachtsheer provides an "IC adjustment" that can be used to convert rank or ordinal valuations into inputs for optimization.[22] The appendix describes a generalization of this process.

The IC adjustment converts forecasts to a scale that represents the average return associated with the forecast. The "adjusted" returns are then on a scale comparable to the risk estimates and other constraints used in the optimization, such as transaction costs. The IC adjustment may operate as a shrinkage operator formally similar to a Bayes-Stein estimator.

While widely used, the procedure is often not well understood. Although the sole purpose of the IC adjustment is to convert forecast returns onto an economically meaningful scale, it is frequently employed as an *ad hoc* method for controlling the optimization results by adjusting the magnitude of input alphas. The IC adjustment as used in practice is thus often fallacious. In particular, as the example in the appendix shows, correct IC adjustment may not even change the size of an alpha derived from a traditional dividend discount model under common assumptions.

Alternatives to MV Optimization

The non-uniqueness of MV-optimal portfolios has important implications for active equity managers. Most importantly, the inherent ambiguity of optimal portfolio structure provides a rationale for choosing from among statistically equivalent optimal portfolios that portfolio most consistent with priors of financial relevance, understandability and marketability.

Understandability and financial meaning are fundamentally important practical considerations in defining valid portfolio construction criteria. Valid financial considerations, not the rigid application of a computer program, should dominate the portfolio construction process.

The Linear Programming Alternative

The non-uniqueness principle allows consideration of linear programming as an alternative optimization technology. Linear programming has been successfully used by a number of organizations to optimize fixed income portfolios for a wide range of purposes. It is also an integral part of the investment process of some active equity managers.[23]

For equity optimization, linear programming provides a tool for designing portfolios with maximum (excess) return with specified financial characteristics, such as specific values or ranges of beta or yield. Simple techniques, which in many cases imitate the activities of active managers, can be used to control the overall level of diversification and extra-market risk in the portfolio. Other constraints can be imposed to take transaction costs and liquidity into account. The end result is an "optimized" portfolio with well-defined risk characteristics and readily understandable structure that avoids important errors associated with the overuse of information in statistical estimates.

Linear programming does not eliminate the problem of error maximization, although the errors may be easier to understand and correct. This is because the error-maximization process itself is linear, not quadratic; high or low concentrations in a stock or sector can easily be traced to large or small estimates of return.

"Optimal" portfolios based on linear programming may be criticized because they are not mean-variance efficient and are therefore subject to the possible misuse of valid forecast information.[24] But such theoretical criticisms have little practical relevance if the "optimal" portfolio structure cannot be unambiguously defined in the context of the statistical limitations of the input data. At the current state of technology, an enhanced linear programming algorithm, carefully defined and controlled, may be useful in providing a practical balance of limitations and benefits, especially for active equity management.

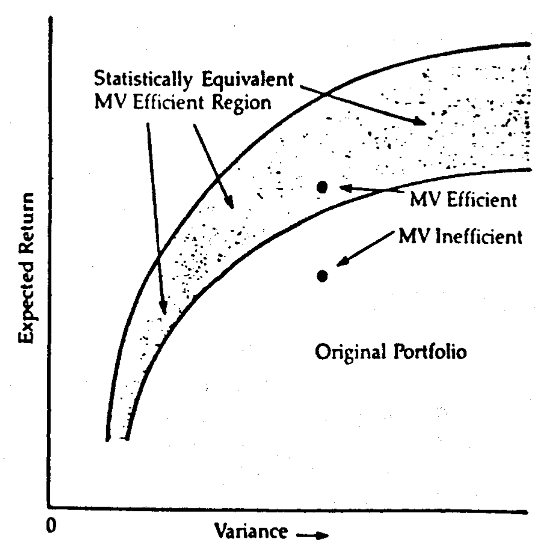
Ultimately, the benefits of equity optimization based on linear programming technology must be judged relative to the benefits that can be provided by carefully defined, input-adjusted, MV optimization. This is still an open issue. In many cases, however, the problems of error maximization that most limit the practical value of MV optimization seem largely attributable to estimate errors in the return, rather than the risk, dimension. This suggests that the value of linear programming technology for practical investment management may ultimately be limited.

Testing for Mean-Variance Efficiency

An important alternative to MV optimization is to test for the MV efficiency of a given portfolio.[25] Figure G illustrates the procedure. The portfolio to be tested is represented by a point below the efficient frontier. The test determines whether or not the given portfolio lies within a confidence region of portfolios that are statistically equivalent to points on the MV frontier. The confidence region increases as expected return increases, reflecting the assumed lower accuracy of the estimates of expected return versus risk.[26]

Such a procedure, at least conceptually, is very attractive. Assuming that the investor or institution considers the portfolio to be tested as "optimal," the portfolio represents a revealed preference about the appropriate level of risk. The issue of which utility function to use to

Figure G Test for MV Efficiency



define a point on the efficient frontier may thus not be material.

There are two substantive problems with the approach. First, the tests have little power; the null hypothesis—MV efficiency—will often not be rejected when it is false. Second, current tests are inappropriate for many problems of practical interest; the MV efficient frontier is unconstrained, including allowance for short sales. As a practical matter, available tests have limited value as indicators of whether a given portfolio is "far" from the (unconstrained) MV efficient frontier.

Specialized Applications of MV Optimization

The substantial limitations of MV optimization when applied to the general problem of optimal portfolio construction have been noted above. Some important specialized applications are discussed below.

Index or Tracking Funds

MV optimizers may be used to structure "index" or "tracking" funds for equity management. The objective is to structure from a prescribed set of securities a portfolio whose performance will be similar (within a specified tolerance) to that of a given index. The optimization is defined by setting the return inputs equal to zero and imposing few, if any, constraints on the solution. In this case, the MV efficient frontier reduces to a single point—the portfolio with minimum tracking error or residual variance.

Most of the problems normally associated with MV optimizers are eliminated, or greatly reduced, when MV optimization is applied to indexing. There is no error maximization resulting from errors in the return estimates. No errors are created by mismatches in the levels of uncertainty in the return versus risk estimates. Because the exact structure of the optimal solution is not the focus of the analysis, neither nonuniqueness nor unintuitiveness is an important consideration. The basic remaining source of error is the adequacy of the risk model. If the index is standard, MV optimization may provide a useful procedure for defining tracking funds. Error maximization may be exhibited primarily in terms of downward-biased estimates of the tracking error.

Nevertheless, two of the most widely used procedures for index fund management—replication and stratified sampling—do not require MV-optimization technology.[27] Replication funds index-weight the stocks in an index fund, with minimal restrictions. Stratified-sampling funds include a small number of large-capitalization securities plus a selection of securities within each industry group to match the capitalization weights of the index. The lesson of practice suggests that the strength of the error-maximization process and/or the limitations of available equity risk models may significantly limit the value of MV optimizers even when applied to structuring pure index funds.

Tilted Index Funds

Tilted index or "value-added" funds minimize tracking error with respect to the selected index while maximizing other portfolio characteristics, such as dividend yield. These additional objectives can be treated as factors in the optimization objective function. As components of a separable utility function, they operate as penalty functions trading off one portfolio attribute for another.[28] This procedure can raise problems, however, both because it is likely to introduce biased estimates of the characteristics included in the objective and because it requires assigning appropriate utility weights to the factor functions, which may be very difficult to rationalize or control.

Asset Allocation

When applied to the asset allocation problem, MV optimization aims to find an optimal mix of asset classes. The analysis may include domestic and foreign equity market indexes and various categories of corporate and government bonds. The number of assets is generally small, usually significantly less than 20.

MV-optimization estimates, even when based primarily on historical data, may be reasonably reliable in the asset-allocation context. This is because a relatively small number of estimates are required and because they are often intended to reflect the long-term structure of financial markets. The character of the "optimal" allocation may consequently be anticipated, and errors created by the input estimates more easily controlled. Benchmark asset allocation may be particularly beneficial in reducing the impact of estimation errors and focusing the optimization process on the investor's valid investment objectives.

On the negative side, asset allocation without a suitable benchmark is significantly error-prone because the focus is on the least reliable result of the optimization—the financial structure of the optimal asset mix. The most important operative issue—non-uniqueness—implies that there may be statistically equivalent MV-optimal asset allocations with very different financial structures. This effect is often observed in the time-period sensitivity of asset-allocation results. A reliable optimal asset mix recommendation requires a more than casual understanding of the characteristics of the confidence region associated with the input estimates.

Conclusion

Markowitz MV–optimization technology is not easy to use properly. In fact, equal weighting may significantly outperform unconstrained MV optimization in many cases. The fundamental problem is that the level of mathematical sophistication of the optimization algorithm is far greater than the level of information in the input forecasts. MV optimizers operate in such a manner that they magnify the errors associated with the input estimates. Without careful problem definition derived from sound investment judgment and sophisticated adjustment of the inputs, MV optimization may often do more harm than good.

Given the increasing potential for error maximization as the number of assets under consideration increases, the optimization process must be carefully controlled if it is to produce solutions of practical value for active equity managers. Optimization should be defined with respect to a suitable benchmark. Transaction and liquidity cost constraints and appropriate adjustment of the inputs are generally recommended. Sector and industry constraints, if available and appropriate, may be usefully included. The operative principle is that any information on the structure of the optimal prior, to the extent that it is reliable, may be included in the definition of the optimization problem.

Nevertheless, the fact that priors and constraints are generally required to derive financially meaningful optimized portfolios may indicate that MV optimization is often superfluous or even counterproductive, and that simpler, more direct approaches to portfolio construction may often be advisable. ■

Appendix

Traditional Quadratic (MV) Optimization

MV optimization maximizes

$$\mu\text{-}\lambda\sigma^2$$

or

$$\alpha\text{-}\lambda\omega^2\text{-}k(\beta\text{-}\beta_T)^2$$

subject to the following linear constraints

$$X \geq 0$$

$$\Sigma X = 1$$

where

μ = expected portfolio return,
σ^2 = portfolio variance,
X = proportion of initial wealth invested in an individual asset,
λ = risk-aversion parameter, which varies to trace out the MV efficient frontier,
α = expected portfolio systematic risk-adjusted (residual) return,
β = estimated portfolio systematic (beta) risk,
ω^2 = estimated portfolio residual risk,
β_T = target portfolio beta, and
k = a prespecified positive constant.

The computation may include transaction cost and other linear constraints (e.g., yield or P/E portfolio values).

The IC Adjustment

A simple use of linear least-squares regression provides a useful foundation for the IC adjustment procedure for many applications. Assume that a forecast process provides estimates of systematic risk-adjusted returns or alphas, where the cross-sectional mean is zero. Assume also that the subsequent ex post alphas for each stock are also available, with mean equal to zero. Perform a linear least-squares regression of ex post against ex ante alphas:

$$A_i = c + d\alpha_i + \varepsilon_i$$

where

A_i = ex post alpha,
c = the constant linear regression parameter,

d = the slope linear regression parameter,
α_i = ex ante alpha and
ε_i = the error term.

By assumption, $c = 0$. By definition, the regression coefficient, d, is the IC adjustment, where

$$d = IC\sigma(A)/\sigma(\alpha)$$

and

IC = cross-sectional "information" correlation of ex ante and ex post alpha,
$\sigma(A)$ = cross-sectional standard deviation of ex post alpha and
$\sigma(\alpha)$ = cross-sectional standard deviation of ex ante alpha.

The product, $d\alpha_i$, is interpretable as the "excess return on average associated with forecast alpha, α_i." The IC adjustment parameter, d, provides the appropriate scale transformation of the forecast alpha with respect to ex post alpha for the given stock universe.

Note that the value of d requires three forecasts for the given stock universe—(1) the IC value; (2) the ex post level of volatility, $\sigma(A)$; and (3) the implicit forecast horizon. The values of IC and $\sigma(A)$ can be estimated from historical data or may be input as subjective estimates. For many stock valuation models, the IC is assumed to have a value of the order 0.05 to 0.20. In applications, it may be appropriate to use different values of the IC adjustment parameters, depending on the characteristics of the stock universe (e.g., growth stocks are likely to have very different IC and volatility values and forecast horizon than utility stocks).

The IC adjustment is essentially a two-step process—a ratio scale transformation of the forecasts, indicated by the ratio $\sigma(A)/\sigma(\alpha)$, followed by a transformation based on the level of the information in the forecasts, indicated by the IC multiplication. It is this first step that is often not well understood in traditional applications of the procedure.

For ordinal or rank data, the Ambachtsheer procedure for creating forecast alpha implies that $\sigma(A) = \sigma(\alpha)$ by construction. In this case, d = IC, which rationalizes common institutional practice. For the traditional dividend discount model (DDM) alpha, the simple IC procedure—multiplying alpha by the value of IC—is often not valid.[29] The problem is: Under what conditions can we assume that $\sigma(A) = \sigma(\alpha)$?

Consider the following exercise. For many traditional DDMs, $\sigma(\alpha)$ is approximately 3 per cent.[30] Assume, as is traditional in applications associated with DDM alphas, a forecast horizon of one year and a representative capital market universe. To compute $\sigma(A)$ assume a market standard deviation of 20 per cent and a multiplier of 1.5 for the cross-sectional standard deviation of stock alpha; i.e., $\sigma(A)$ is approximately 30 per cent. Finally, assume an IC value of 0.1. In this case, the valid IC adjustment of DDM alpha is one; the IC adjustment is not a shrinkage operator. Valid shrinkage of DDM alphas may require a shorter forecast horizon assumption and/or lower IC value.

Footnotes

1. H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments* (New York: John Wiley, 1959).
2. W. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, September 1964, and J. Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, February 1965.
3. W. Gray, "Portfolio Construction: Equity," in J. Maginn and D. Tuttle, eds., *Managing Investment Portfolios: A Dynamic Approach* (Boston: Warren, Gorham & Lamont, 1983), p. 415.
4. N. Elton, M. Gruber and M. Padberg, "Optimal Portfolios from Simple Ranking Devices," *Journal of Portfolio Management*, Spring 1978.
5. J.D. Jobson and B. Korkie, "Putting Markowitz Theory to Work," *Journal of Portfolio Management*, Summer 1981.
6. The scope of this article is limited to the assumptions implicit in the valid application of the MV efficient frontier framework. It does not address (1) time horizon and multiperiod MV efficiency, (2) the appropriate theoretical framework for defining systematic risk, or (3) optimization in the sense of strategic asset allocation with respect to the business risks of the firm or liabilities of the fund. For (1) see, for example, Markowitz, *Portfolio Selection, op. cit.*, Chapter 6; H. Latane, "Criteria for Choice Among Risky Ventures," *Journal of Political Economy*, April 1959; N. Hakansson, "A Characterization of Optimal Multi-Period Portfolio Policies," in N. Elton and M. Gruber, eds., *Portfolio Theory, 25 Years After* (New York: North Holland, 1979); and R. Michaud, "Risk Policy and Long-Term Investment," *Journal of Financial and*

Quantitative Analysis, June 1981. For (2) see S. Ross, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, December 1976 and "Return, Risk and Arbitrage," in I. Friend and J. Bicksler, eds., *Risk and Return in Finance* (Cambridge, MA: Ballinger, 1977). For (3) see R. Roll and S. Ross, "The Arbitrage Pricing Theory Approach to Strategic Portfolio Planning," *Financial Analysts Journal*, May/June 1984.

7. See A. Rudd and B. Rosenberg, "Realistic Portfolio Optimization," in Elton and Gruber, eds., *Portfolio Theory, op. cit.* Equity residual return or alpha is the difference between total return and the total return associated with the level of systematic risk. Residual risk is the variance of residual return, which is parametric in the level of systematic risk.
8. W. Sharpe, *Investments*, 3rd ed. (Englewood Cliffs, NJ: Prentice-Hall, 1985), pp. 666–670.
9. J.D. Jobson and B. Korkie, "Estimation for Markowitz Efficient Portfolios," *Journal of the American Statistical Association*, September 1980 and Jobson and Korkie, "Putting Markowitz Theory to Work," *op. cit.*
10. For a recent discussion, see P. Frost and J. Savarino, "For Better Performance: Constrain Portfolio Weights," *Journal of Portfolio Management*, Fall 1988.
11. Admissibility is a minimum condition used to reduce the decision problem without loss of relevant information. See E. Lehmann, *Testing Statistical Hypotheses* (New York: John Wiley, 1959), p. 16.
12. C. Stein, "Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution," *Proceedings of the 3rd Berkeley Symposium on Probability and Statistics* (Berkeley: University of California Press, 1955). For example, vector of means, independent, normally distributed, quadratic loss functions, known covariance matrix.
13. P. Jorion, "International Portfolio Diversification with Estimation Risk," *Journal of Business*, July 1985.
14. The shaded region below the efficient frontier represents a rough illustration of a confidence region, based on some unpublished simulations provided by P. Jorion.
15. See P. Samuelson, "Computer Construction of Optimal Portfolios," in S. Levine, ed., *The Investment Manager's Handbook* (Homewood, IL: Dow Jones-Irwin, 1980) and W. Sharpe, "Microcomputer Perspectives: Asset Allocation Systems," *Financial Analysts Journal*, May/June 1985.
16. A. Perold, "Large Scale Portfolio Optimization," *Management Science*, October 1984.
17. D. Seidel, "Market Indices in the Asset Allocation Framework" (presented to the Institute for Quantitative Research in Finance, Spring 1986).
18. See R. Michaud, "Pension Policy and Benchmark Optimization," *Pension Executive Review* (Merrill Lynch, New York, July 1988) for a description of the procedure.
19. See Jorion, "International Portfolio Diversification," *op. cit.* and "Bayes-Stein Estimation for Portfolio Analysis," *Journal of Financial and Quantitative Analysis*," September 1986.
20. Jobson, "Putting Markowitz Theory to Work," *op. cit.* and Jorion, "International Portfolio Diversification" and "Bayes-Stein Estimation," *op. cit.*
21. *Ibid.*
22. K. Ambachtsheer, "Where Are the Customers' Alphas?" *Journal of Portfolio Management*, Fall 1977.
23. J. Farrell, *Guide to Portfolio Management* (New York: McGraw-Hill, 1983), pp. 168–174.
24. See footnote 8.
25. See, for example, J.D. Jobson and B. Korkie, "Potential Performance and Tests of Portfolio Efficiency," *Journal of Financial Economics*, September 1982; J. Shaken, "Multivariate Tests of the Zero-Beta CAPM," *Journal of Financial Economics*, September 1985 and "Testing Portfolio Efficiency when the Zero Beta Rate is Unknown," *Journal of Finance*, 1986, pp. 269–276; R. Roll, "A Note on the Geometry of Shaken's CSR T^2 Tests for Mean/Variance Efficiency," *Journal of Financial Economics*, September 1985; and C.E. Amsler and P. Schmidt, "A Monte Carlo Investigation of the Accuracy of Multivariate CAPM Tests," *Journal of Financial Economics*, 1985, pp. 359–375.
26. I am indebted to P. Jorion for this observation.
27. For example, Wilshire Associates' index fund management service uses index replication and stratified sampling exclusively.
28. See Rudd and Rosenberg, "Realistic Portfolio Optimization," *op. cit.*
29. Generally defined as the residual formed from the linear least-squares regression of DDM implied returns against beta.
30. See R. Michaud and P. Davis, "Valuation Model Bias and the Scale Structure of Dividend Discount Returns," *Journal of Finance*, May 1982.